Pearcey Kernel Basis: Beyond Second Order Differential Eigenvalue Equations

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The *Pearcey kernel* was introduced by Brézin and Hikami. It describes the eigenvalues of a large random Hermitian matrix shifted by a fixed matrix with eigenvalues $\pm a$, near the point where the two clusters of eigenvalues merge. We build a basis for this kernel following the Gaussian unitary ensemble, whose kernel is the projector onto the first N Hermite functions.

For the shifted matrix at finite N, the kernel has no decomposition into an orthonormal basis. It does have a biorthogonal one: two families of functions, dual to each other, in which the kernel is the sum of the first $2N$ terms. In the scaling limit this basis becomes a continuous family indexed by $\lambda \in \mathbb{R}$, and K^{Pear} is its integral over $\lambda > 0$, as the Airy kernel is the integral of the shifted Airy functions over the half-line. At finite N the proofs are complete; in the limit some steps are formal.

We assume familiarity with the Fourier transform, contour integrals and residues, linear differential equations and the Dirac delta function. Section 1 is a short introduction to random matrices: the Gaussian unitary ensemble, determinantal point processes, and the sine and other kernels. Readers who know them can skim it, but the later sections follow its pattern and refer to its equations.

1 Crash Introduction

1.1 Gaussian Unitary Ensemble

Let H be an $N \times N$ Hermitian matrix. We define a probability measure on such matrices by the density

$$\rho_N^{\mathrm{GUE}}(H) := Z_N^{-1} \, \mathrm{e}^{-\frac{1}{2}\operatorname{Tr} H^2}, \tag{1}$$

with respect to the Lebesgue measure on the N^2 real parameters of H: the diagonal entries and the real and imaginary parts of the entries above the diagonal. Here Z_N is a normalization constant that depends only on N and may differ from one equation to another. This model is called the *Gaussian unitary ensemble* (GUE), because the density is invariant under the transformation $H \to U^{-1}HU$ for any unitary matrix U.

The eigenvalues $\{\lambda_i\}_{0 \le i < N}$ of H then have the joint density (see, for example, M. L. Mehta, *Random Matrices*, 3rd ed., Elsevier, 2004, the chapter on the joint probability density function for the eigenvalues)

$$\rho_N^{\mathrm{GUE}}(\lambda_0, \lambda_1, \dots \lambda_{N-1}) = Z_N^{-1} \prod_{0 \le i < j < N} |\lambda_i - \lambda_j|^2 \prod_{0 \le i < N} \mathrm{e}^{-\frac{\lambda_i^2}{2}} = Z_N^{-1} \, \mathrm{e}^{2\sum_{0 \le i < j < N} \log|\lambda_i - \lambda_j| - \frac{1}{2}\sum_{0 \le i < N} \lambda_i^2}. \tag{2}$$

This density is a Boltzmann distribution for a one-dimensional gas of N particles in an external quadratic potential, with a logarithmic repulsion between each pair of particles, the pair energy being $E_{ij} = -2\log|\lambda_i - \lambda_j|$.

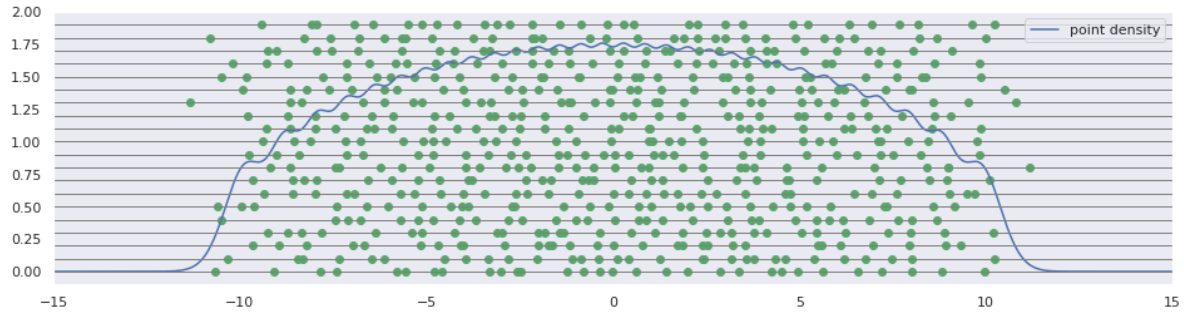

Figure 1: Samples of eigenvalues λ for $N = 30$. Each of the 20 rows is one random sample of the N eigenvalues (particles), shown in green. The blue curve is the mean eigenvalue density.

1.2 Determinantal Point Process

The eigenvalue distribution of the previous subsection is an example of a *determinantal point process*, a random collection of N unordered real numbers $\{\lambda_i\}_{0\le i<N}$ with a joint density of the form

$$\rho_N(\lambda_0, \lambda_1, \dots \lambda_{N-1}) := Z_N^{-1} \det\left[\left\{K(\lambda_i, \lambda_j)\right\}_{0\le i,j<N}\right], \tag{3}$$

where $K(x, y)$ is a real-valued function of two arguments. We read it as the kernel of the linear operator

$$K : f(x) \mapsto (Kf)(x) := \int \mathrm{d}y\ K(x, y) f(y), \tag{4}$$

acting on functions $f(x)$ of one real variable. We assume that K has the projection property

$$\int_{-\infty}^{+\infty} \mathrm{d}y\ K(x, y)K(y, z) = K(x, z), \tag{5}$$

and that its trace is the number of particles:

$$\int_{-\infty}^{+\infty} \mathrm{d}x\ K(x, x) = N. \tag{6}$$

Under these two assumptions, the correlation functions of any L of the particles have the same determinantal form as ρ_N (Gaudin's lemma, see Mehta, Random Matrices):

$$R_{N,L}\left(\lambda_0, \lambda_1, \dots \lambda_{L-1}\right) = \det\left[\left\{K(\lambda_i, \lambda_j)\right\}_{0\le i,j<L}\right], \quad L = 1, 2, \dots N \tag{7}$$

In particular, the first correlation function

$$R_{N,1}(\lambda) = K(\lambda, \lambda) \tag{8}$$

gives the mean density of the eigenvalues. The two-point correlation function is

$$R_{N,2}(\lambda_0, \lambda_1) = K(\lambda_0, \lambda_0)K(\lambda_1, \lambda_1) - K(\lambda_0, \lambda_1)^2. \tag{9}$$

The eigenvalue distribution of the previous subsection is a determinantal point process with the kernel (see, for example, Mehta, Random Matrices)

$$K_N^{\mathrm{GUE}}(x, y) = \sum_{0 \le i < N} \psi_i(x)\psi_i(y). \tag{10}$$

Here $\psi_i(x)$ are the Hermite functions

$$\psi_i(x) := \frac{1}{(2\pi)^{\frac{1}{4}} i!^{\frac{1}{2}}} \mathrm{e}^{-\frac{x^2}{4}} \mathrm{He}_i(x), \quad i = 0, 1, 2, \dots , \tag{11}$$

where $\mathrm{He}_i(x)$ is the Hermite polynomial of order i. The Hermite functions form an orthonormal basis of $L_2(\mathbb{R})$: the family is orthonormal,

$$\int_{-\infty}^{+\infty} \mathrm{d}x \ \psi_i(x)\psi_j(x) = \delta_{i,j}, \quad i, j = 0, 1, 2, \dots \tag{12}$$

and complete,

$$\sum_{i=0}^{+\infty} \psi_i(x)\psi_i(y) = \delta(x - y). \tag{13}$$

The identity (13) says that the integral operator with the kernel on its left-hand side is the identity operator: for every $f \in L_2(\mathbb{R})$,

$$f(x) = \sum_{i=0}^{+\infty} \psi_i(x) \int_{-\infty}^{+\infty} \mathrm{d}y \ \psi_i(y) f(y), \tag{14}$$

with the series converging in $L_2(\mathbb{R})$. For f in the Schwartz space, the smooth functions that decrease faster than any power of x together with all their derivatives, the series also converges uniformly in x. In this sense $\delta(x - y)$ is the kernel of the identity operator, and the delta functional takes f to $f(0)$. Every identity with δ below is read the same way, after integration against a test function f, which we drop for brevity. The class of test functions matters: for the continuous basis of Section 2.3.2 the Schwartz space is too large, and we use a narrower class there.

So the eigenvalue distribution, through its kernel $K_N^{\mathrm{GUE}}(x, y)$, is tied to the orthonormal basis $\{\psi_i(x)\}_{i=0}^{+\infty}$.

The Hermite functions also solve a second-order differential equation. For $i \in \mathbb{R}$, the equation

$$\psi''(x) + \left(i + \frac{1}{2} - \frac{x^2}{4} \right) \psi(x) = 0 \tag{15}$$

has a nonzero solution square-integrable on $\mathbb{R}$ only for $i = 0, 1, 2, ...$, and every such solution is a constant multiple of ψ_i.

For real symmetric matrices with the same density (the *Gaussian orthogonal ensemble*), the factor $|\lambda_i - \lambda_j|^2$ above becomes $|\lambda_i - \lambda_j|$, and the eigenvalues form a Pfaffian point process rather than a determinantal one. We do not use it below.

To summarize this subsection, we have defined a family of distributions of $N = 0, 1, 2, ...$ real numbers. Each of them comes from the projection operator K_N^{GUE} onto an N-dimensional subspace $V_N^{\mathrm{GUE}} \subset L_2(\mathbb{R})$, and a single orthonormal basis $\{\psi_i\}_{i=0}^{+\infty}$ gives all these subspaces at once:

$$V_N^{\mathrm{GUE}} = \mathrm{span}\{\psi_i\}_{i=0}^{N-1}, \quad N = 0, 1, 2, ... \ . \tag{16}$$

1.3 Scaling Limit

In statistical physics and elsewhere, we often take the limit $N \to +\infty$, where N is the number of particles, the dimension, or another discrete parameter. The limiting model can show remarkable behavior. We look at two examples below and then return to the GUE.

One of the simplest examples is the central limit theorem. In particular, it states that if the independent, identically distributed random variables $\{X_i\}_{0 \le i < N}$ have mean 0 and variance 1, then their sum divided by $\sqrt{N}$ approaches the standard normal distribution as $N \to +\infty$:

$$\mathrm{law}\left[\frac{\sum_{0 \le i < N} X_i}{\sqrt{N}}\right] \quad \xrightarrow{N \to \infty} \quad \mathcal{N}(0, 1). \tag{17}$$

The arrow means convergence in distribution: the distribution functions converge at every point.

We note three features of this limit:

1. The limit law exists after a rescaling, here by the factor $N^{-\frac{1}{2}}$.
2. The limit law is universal: whatever the law of X_i with mean 0 and variance 1, the limit is the normal law.
3. The limit law has a symmetry that the initial laws may lack. For example, the normal law is symmetric under the reflection $x \to -x$, while the law of a finite sum of the X_i need not be.

Another classical example of a *scaling limit* is the *Ising model*, a probability model on a finite lattice, say a square one of size $N \times N$. As $N \to +\infty$, the model has a phase transition in the temperature. At the critical temperature, with the lattice spacing taken as $1/N$, the limit law is invariant under angle-preserving maps and does not depend on the lattice type, within a wide class of planar lattices. Both statements are theorems for the main observables of the model; see Chelkak and Smirnov and Chelkak, Hongler and Izyurov.

Much of the interest of the random matrix model of Section 1.1 lies in the limit as N tends to infinity, in the eigenvalues of an "infinite random matrix".

1.4 Sine Process

First, note that if $\{\psi_i(x)\}_{i=0}^{+\infty}$ is an orthonormal basis of $L_2(\mathbb{R})$, then

$$\{|A|^{\frac{1}{2}}\psi_i(Ax+B)\}_{i=0}^{+\infty}, \quad A \neq 0, \quad B \in \mathbb{R}, \tag{18}$$

is also an orthonormal basis.

To find a limit law for the eigenvalues of the GUE, we take the limit of its kernel:

$$C_N \cdot K_N^{\mathrm{GUE}}(\pi N^{-\frac{1}{2}}x, \pi N^{-\frac{1}{2}}y) \quad \xrightarrow{N\to\infty} \quad K^{\mathrm{Sine}}(x,y) := \frac{\sin \pi(x-y)}{\pi(x-y)}. \tag{19}$$

Here

$$C_N = \pi N^{-\frac{1}{2}} \tag{20}$$

is the Jacobian of the substitution, so $C_N K_N^{\mathrm{GUE}}$ is the kernel of the rescaled eigenvalues $\pi^{-1}N^{\frac{1}{2}}\lambda_i$. The factor π makes the mean spacing equal to 1 near the origin, since the eigenvalue density of the GUE at 0 is $\pi^{-1}N^{\frac{1}{2}}$ to leading order, so $K^{\mathrm{Sine}}(0,0) = 1$. The alternative choice $C_N = 1/K_N^{\mathrm{GUE}}(0,0)$ differs from this one by a factor tending to 1. The limit K^{Sine} is the *sine kernel*. It defines a point process, a random infinite countable set of real numbers, whose correlation functions are given by (7) with this kernel.

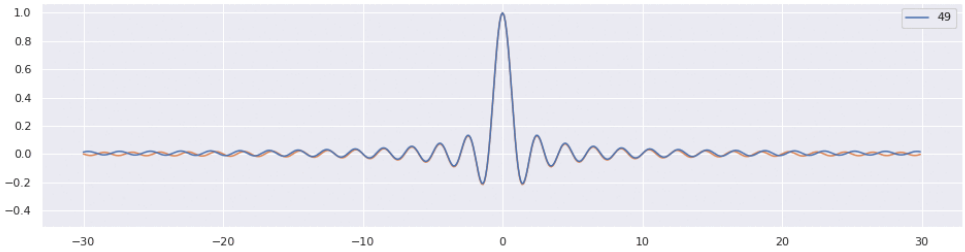

Figure 2: An illustration of limit (19) for different values of $x-y$ with the restriction $x+y=0$. By (9), the two-point correlation function of the limit process is $R_2^{\mathrm{Sine}}(x,y) = 1 - K^{\mathrm{Sine}}(x,y)^2$, a function of the difference $x-y$.

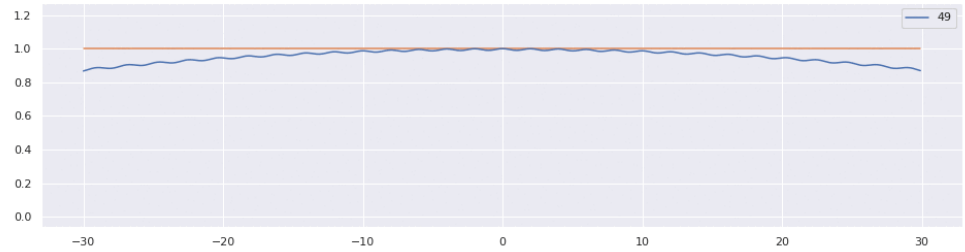

Figure 3: An illustration of limit (19) on the diagonal $x = y$, as a function of x. On the diagonal the kernel is the mean density of the eigenvalues λ.

The sine process is invariant under the translations $x \to x + c$ for any real c, while its finite-N analogues are not. Physically, the sine process is a one-dimensional gas of particles with logarithmic repulsion, spread uniformly over the whole real line with mean density 1.

To understand this limit, note that for any $\omega > 0$

$$\frac{\psi_i\left(\pi N^{-\frac{1}{2}} x\right)}{\psi_i(0)} \xrightarrow[\frac{i}{N} \to\, \omega^2]{N,\, i \to\, \infty} \cos(\pi\omega x), \quad i = 2k, \quad k = 0, 1, 2, \dots\,, \tag{21}$$

$$\frac{\psi_i\left(\pi N^{-\frac{1}{2}} x\right)}{i^{-\frac{1}{2}}\psi_i'(0)} \xrightarrow[\frac{i}{N} \to\, \omega^2]{N,\, i \to\, \infty} \sin(\pi\omega x), \quad i = 2k+1, \quad k = 0, 1, 2, \dots \tag{22}$$

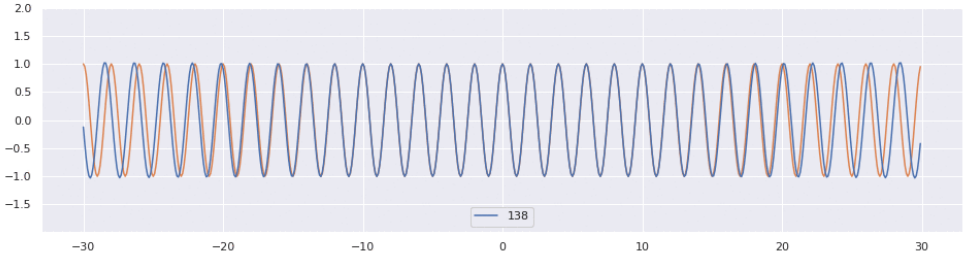

Figure 4: Illustration of the cosine limit above for $\omega = 1$.

Now decompose both sides of limit (19):

$$K_N^{\mathrm{GUE}}(\pi N^{-\frac{1}{2}} x, \pi N^{-\frac{1}{2}} y) = \sum_{0 \le i < N} \psi_i\left(\pi N^{-\frac{1}{2}} x\right) \psi_i\left(\pi N^{-\frac{1}{2}} y\right). \tag{23}$$

$$K^{\mathrm{Sine}}(x, y) = \int_0^1 \mathrm{d}\omega \;\; (\cos(\pi\omega x)\cos(\pi\omega y) + \sin(\pi\omega x)\sin(\pi\omega y))\,. \tag{24}$$

By (22), the rescaled even and odd Hermite functions tend to cos and sin respectively, up to normalization. As $N \to +\infty$, the values $\left\{\sqrt{\frac{i}{N}}\right\}_{0 \le i < N}$ fill the interval $[0,1]$ densely, and this interval is the range of ω in the integral for K^{Sine} above. So, heuristically, the decomposition over the discrete spectrum at finite N approaches the decomposition over the continuous Fourier spectrum.

The Fourier basis is not a basis in the sense of the Hermite functions: it is not a countable family of square-integrable functions. A rigorous definition needs a *rigged Hilbert space*; see I. M. Gelfand and N. Ya. Vilenkin, *Generalized Functions*, Vol. 4: *Applications of Harmonic Analysis*, Academic Press, 1964.

Finally, consider the differential equation (15). The rescaled function $f(x) := \psi_i\left(\pi N^{-\frac{1}{2}} x\right)$ satisfies

$$\frac{N}{\pi^2} f''(x) + \left(i + \frac{1}{2} - \frac{\pi^2 x^2}{4N}\right) f(x) = 0. \tag{25}$$

Multiplying by $\frac{\pi^2}{N}$ and letting $N \to +\infty$ with $\frac{i}{N} \to \omega^2$, we obtain the differential equation for $\cos(\pi\omega x)$ and $\sin(\pi\omega x)$:

$$f''(x) + \pi^2 \omega^2 f(x) = 0. \tag{26}$$

So the scaling limit to the sine process shows up in at least four ways:

1. Limit of the distributions of the points λ.
2. Limit of the kernels $K(x,y)$.
3. Limit of the subspaces of $L_2(\mathbb{R})$ onto which the operators K project. These subspaces can be described by a suitably rescaled orthonormal basis $\{\psi_i\}_{i=0}^{+\infty}$.
4. Limit of the differential equations.

The sine limit is not the only possibility. For example, a similar limit for the substitution $x \to 2N^{\frac{1}{2}} + N^{-\frac{1}{6}} x$ leads to the *Airy kernel*:

$$K^{\text{Airy}}(x,y) := \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x-y} = \int_0^{+\infty} \mathrm{d}\lambda\ \text{Ai}(\lambda + x)\text{Ai}(\lambda + y), \tag{27}$$

where Ai is the Airy function, whose shifts form an orthogonal basis in the sense that

$$\int_{-\infty}^{+\infty} \mathrm{d}\lambda\ \text{Ai}(\lambda + x)\text{Ai}(\lambda + y) = \delta(x-y). \tag{28}$$

Another example comes from the Laguerre polynomials, orthogonal on the half-axis $(0, +\infty)$, and leads to the *Bessel kernel* (see Tracy and Widom):

$$K^{\text{Bessel}}(x,y) := \frac{J_\alpha(x^{\frac{1}{2}}) y^{\frac{1}{2}} J'_\alpha(y^{\frac{1}{2}}) - x^{\frac{1}{2}} J'_\alpha(x^{\frac{1}{2}}) J_\alpha(y^{\frac{1}{2}})}{2(x-y)}, \quad x, y > 0 \tag{29}$$

Almost all such limits can be understood in terms of orthonormal bases, linear subspaces of L_2, and the well-studied second-order differential equations behind them. The model of the next section does not fit this structure.

2 Pearcey Kernel

2.1 Biased Unitary Ensemble

Brézin and Hikami considered the following random matrix model, the GUE with an external source, which we call the biased unitary ensemble. Let H be a $2N \times 2N$ Hermitian matrix, and let H_a be a fixed $2N \times 2N$ Hermitian matrix with only two distinct eigenvalues, a and $-a$ (with $a > 0$), each of multiplicity N.

The distribution of H has the density

$$\rho_N^a(H) := (Z_N^a)^{-1} \, \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}(H+H_a)^2}, \tag{30}$$

where Z_N^a is a normalization constant. The eigenvalues of H then form a determinantal point process with the kernel

$$K_N^a(x,y) := \frac{-1}{(2\pi)^2 \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}t \oint_{\pm a} \mathrm{d}u \left(\frac{a^2+t^2}{a^2-u^2}\right)^N \frac{1}{u-\mathrm{i}t} \, \mathrm{e}^{-\frac{t^2}{2}-\frac{u^2}{2}-\mathrm{i}tx+uy}. \tag{31}$$

Here $\oint_{\pm a} \mathrm{d}u$ denotes the Cauchy integral in the complex plane, counterclockwise around the poles $u = a$ and $u = -a$.

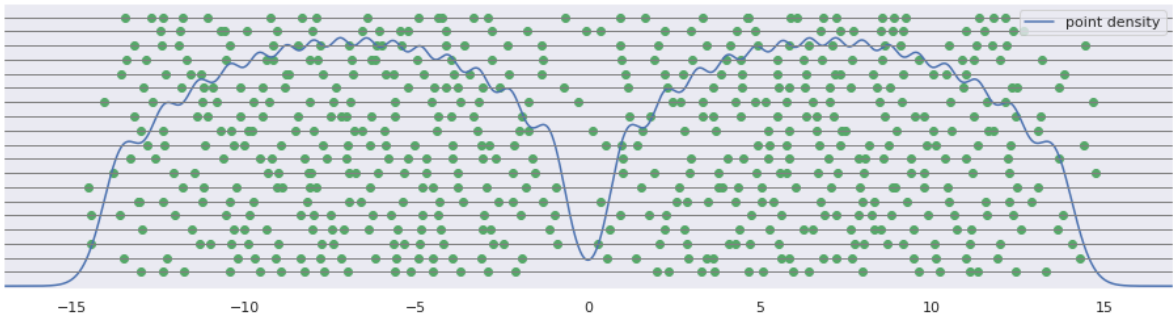

Figure 5: Samples of eigenvalues λ for $N = 15$ and $a = 6$. Each of the 20 rows is one random sample of the $15 \times 2 = 30$ eigenvalues (particles), shown in green. The eigenvalues form two clusters on the real line that are barely joined.

The kernel K_N^a is not symmetric in its arguments. A transformation

$$K(x,y) \to \frac{f(x)}{f(y)} K(x,y) \tag{32}$$

with any function f preserves the determinantal law, but no kernel of this process, obtained by such a transformation or in any other way, is real symmetric or Hermitian. For any kernel K of a determinantal point process,

$$R_{N,2}(x,y) - R_{N,1}(x)\,R_{N,1}(y) = -K(x,y)\,K(y,x), \tag{33}$$

and the left-hand side depends on the process alone. A real symmetric or Hermitian kernel makes it nonpositive. For K_N^a it is positive at some points: eigenvalues in the two clusters are slightly positively correlated. For example, for $N = 1$, $a = 2$, $x = -3$ and $y = 1.2$, we find numerically $R_{N,2}(x,y) - R_{N,1}(x)\,R_{N,1}(y) \approx 0.003$. Hence a decomposition into an orthogonal basis, as in (10), is impossible.

The projection property (5) still holds, so there is a linear space of functions that the integral operator K_N^a leaves unchanged.

2.2 Pearcey Scaling Limit

For small a, the distribution is close to the one of Section 1.1, with all eigenvalues in a single cluster. As a grows, the eigenvalues split into two clusters that separate more and more. Following Brézin and Hikami, we set $a = (2N)^{\frac{1}{2}}$, rescale $x \to (2N)^{-\frac{1}{4}}x$ and let $N \to +\infty$. At this value of a the two clusters are exactly at the transition between merged and separated. The limit

$$K^{\text{Pear}}(x,y) := \lim_{N\to+\infty} (2N)^{-\frac{1}{4}} K_N^{(2N)^{\frac{1}{2}}}\left((2N)^{-\frac{1}{4}}x, (2N)^{-\frac{1}{4}}y\right) \tag{34}$$

is called the *Pearcey kernel*. The factor $(2N)^{-\frac{1}{4}}$ is the Jacobian of the rescaling. The limit exists for all real x, y, as Bleher and Kuijlaars prove (Theorem 1.1 with $b = 0$, for matrices of size $2N$ rescaled by $(2N)^{-\frac{1}{2}}$). Tracy and Widom prove that the convergence to K^{Pear} is uniform on bounded sets in a model of non-intersecting Brownian motions, whose positions at a fixed time have, up to a constant scale factor, the law of the eigenvalues here.

The Pearcey kernel can be written in terms of two functions ϕ and ψ (Brézin and Hikami, eqs. (3.7), (3.13) and (3.21)):

$$K^{\text{Pear}}(x,y) = \frac{\phi'(x)\psi'(y) - \phi''(x)\psi(y) - \phi(x)\psi''(y)}{x-y}, \tag{35}$$

where

$$\phi(x) := \frac{1}{\pi}\int_0^{+\infty} \mathrm{d}t\ \mathrm{e}^{-\frac{t^4}{4}}\cos(tx), \tag{36}$$

$$\psi(x) := -\frac{1}{\pi}\mathrm{Im}\left[\omega\int_0^{+\infty} \mathrm{d}u\,\mathrm{e}^{-\frac{u^4}{4}}\left(\mathrm{e}^{\omega x u} - \mathrm{e}^{-\omega x u}\right)\right], \quad \omega := \mathrm{e}^{\frac{\mathrm{i}\pi}{4}}. \tag{37}$$

Continuity on the diagonal

The formula for K^{Pear} makes sense only for $x \neq y$, and we extend K^{Pear} to the diagonal $x = y$ by continuity. The extension exists and is a smooth function on the whole plane, because the numerator

$$N(x, y) := \phi'(x)\psi'(y) - \phi''(x)\psi(y) - \phi(x)\psi''(y) \tag{38}$$

is smooth and vanishes on the diagonal. It is smooth because the integrals defining ϕ and ψ converge absolutely, together with all their derivatives in x, locally uniformly in x. It vanishes on the diagonal by Brézin and Hikami (Appendix B, eqs. (B8)–(B9)). The value on the diagonal,

$$K^{\text{Pear}}(x, x) = \partial_x N(x, y)\big|_{y=x} = \phi''(x)\psi'(x) - \phi'(x)\psi''(x) - x\,\phi(x)\psi(x), \tag{39}$$

is the density of the Pearcey process (Brézin and Hikami, eq. (3.22)).

The Pearcey kernel is not symmetric, and it contains second derivatives of the functions ϕ and ψ.

2.3 Pearcey Basis

This section builds a basis for the Pearcey kernel, first at finite N and then in the scaling limit.

2.3.1 Finite N Basis

As Section 2.1 shows, the kernel K_N^a has no decomposition into an orthonormal basis. We can still work with a biorthogonal basis, a relaxed version of an orthonormal one. It consists of two families of functions on $\mathbb{R}$ instead of one, $\{e_i\}_{i=0}^{+\infty}$ and $\{e_i^\dagger\}_{i=0}^{+\infty}$, with

$$\int_{-\infty}^{+\infty} \mathrm{d}x\; e_i^\dagger(x) e_j(x) = \delta_{i,j}, \quad i, j = 0, 1, 2, \ldots \tag{40}$$

and

$$\sum_{i=0}^{\infty} e_i(x) e_i^\dagger(y) = \delta(x - y), \tag{41}$$

instead of (12) and (13). This is less restrictive: after the transformation

$$e_i^\dagger(x),\; e_i(x) \to f(x) \cdot e_i^\dagger(x),\; \frac{e_i(x)}{f(x)} \tag{42}$$

we still have a biorthogonal basis, for any function f that is nonzero almost everywhere on $\mathbb{R}$. For a real orthonormal basis, where $e_i^\dagger = e_i$, the same transformation keeps the basis orthonormal only if $f(x)^2 = 1$ almost everywhere.

To build the basis, we first define the functions

$$\phi_n^a(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}t \; (t^2 + a^2)^n \mathrm{e}^{-\frac{t^2}{2} - \mathrm{i}tx}, \quad n = 0, 1, 2, 3, \ldots \,, \tag{43}$$

$$\psi_n^a(x) := \frac{1}{\sqrt{2}\pi\mathrm{i}} \oint_{\pm a} \mathrm{d}u \; \frac{\mathrm{e}^{-\frac{u^2}{2} + ux}}{(a^2 - u^2)^{n+1}}, \quad n = 0, 1, 2, 3, \ldots \,. \tag{44}$$

They are closely related to the multiple Hermite polynomials of the external source model (Bleher and Kuijlaars, 2005); the note at the end of this subsection gives the exact correspondence. These functions solve the *third-order* differential eigenvalue equations

$$(a^2 - \partial_x^2)(\partial_x + x)\phi_n^a(x) = 2n \; \partial_x \phi_n^a(x), \quad n = 0, 1, 2, \ldots \,, \tag{45}$$

and

$$(\partial_x - x)(a^2 - \partial_x^2)\psi_n^a(x) = 2n \; \partial_x \psi_n^a(x), \quad n = 0, 1, 2, \ldots \tag{46}$$

respectively. For example, for $n = 0, 1$

$$\begin{gathered} \phi_0^a(x) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\frac{x^2}{2}}, \quad \phi_1^a(x) = \frac{1}{\sqrt{\pi}} (1 + a^2 - x^2) \mathrm{e}^{-\frac{x^2}{2}}, \\ \psi_0^a(x) = -\sqrt{2} \frac{\mathrm{e}^{-\frac{a^2}{2}}}{a} \sinh(ax), \quad \psi_1^a(x) = \frac{\mathrm{e}^{-\frac{a^2}{2}}}{\sqrt{2} a^3} \left(ax \cosh(ax) - (1 + a^2) \sinh(ax) \right). \end{gathered} \tag{47}$$

We define the biorthogonal basis by

$$e_{2n}^a(x) := 2^{-\frac{1}{2}} \phi_n^a(x), \quad e_{2n+1}^a(x) := 2^{-\frac{1}{2}} \phi_n^{a\,\prime}(x), \quad n = 0, 1, 2, 3, \ldots \,, \tag{48}$$

$$e^{\dagger a}_{2n}(x) := -2^{-\frac{1}{2}} \psi_n^{a\,\prime}(x), \quad e^{\dagger a}_{2n+1}(x) := 2^{-\frac{1}{2}} \psi_n^a(x), \quad n = 0, 1, 2, 3, \ldots \,. \tag{49}$$

The biorthogonality has a structural reason. Let $\mathcal{L} := (a^2 - \partial_x^2)(\partial_x + x)$ and $\mathcal{D} := \partial_x$, with formal adjoints $\mathcal{L}^* = (x - \partial_x)(a^2 - \partial_x^2)$ and $\mathcal{D}^* = -\partial_x$. Then the two eigenvalue equations above read

$$\mathcal{L}\phi_n^a = 2n \, \mathcal{D}\phi_n^a, \quad \mathcal{L}^*\psi_n^a = 2n \, \mathcal{D}^*\psi_n^a, \tag{50}$$

so ϕ_n^a and ψ_n^a are the right and left eigenfunctions of the operator pencil $\mathcal{L} - \nu\mathcal{D}$ for the same eigenvalue $\nu = 2n$. Integrating by parts, which is allowed because ϕ_n^a decays like a Gaussian while ψ_n^a grows at most exponentially, we get

$$2n \int_{-\infty}^{\infty} \mathrm{d}x \; \psi_m^a \mathcal{D}\phi_n^a = \int_{-\infty}^{\infty} \mathrm{d}x \; \psi_m^a \mathcal{L}\phi_n^a = \int_{-\infty}^{\infty} \mathrm{d}x \; (\mathcal{L}^*\psi_m^a) \, \phi_n^a = 2m \int_{-\infty}^{\infty} \mathrm{d}x \; \psi_m^a \mathcal{D}\phi_n^a, \tag{51}$$

hence $\int \mathrm{d}x\ \psi_m^a \phi_n^{a\prime} = 0$ for $m \neq n$. The remaining pairings vanish by parity, since ϕ_n^a is even and ψ_n^a is odd. In the Hermite case a single symmetric operator gives orthogonality. Here the operators are not formally self-adjoint, and the eigenfunctions of the pencil and of its adjoint form two different families. Each eigenvalue $2n$ contributes two basis functions, the even ϕ_n^a and the odd $\mathcal{D}\phi_n^a$, with the duals $\mathcal{D}^*\psi_n^a$ and ψ_n^a of the same parity; the proof below computes their normalization. The limit equations of Section 2.3.2 have the same form with $\mathcal{L} = \partial_x^3 - x$, but there this argument is only formal, for the reason given there.

The real Fourier basis also pairs an even and an odd function, $\cos(\lambda x)$ and $\sin(\lambda x)$, with $\partial_x \cos(\lambda x) = -\lambda \sin(\lambda x)$. There both are eigenfunctions of ∂_x^2 with the eigenvalue $-\lambda^2$; here only ϕ_n^a is an eigenfunction of the pencil, and $\mathcal{D}\phi_n^a$ is not.

In the completeness relation we read $\delta(x-y)$ as for (13), after integration against a test function f. Here the test functions are $f(x) = p(x)\,\mathrm{e}^{-\frac{x^2}{2}}$ with a polynomial p. They are dense in $L_2(\mathbb{R})$, for them the series has finitely many nonzero terms, and the relation holds exactly. We do not claim it for a general $f \in L_2(\mathbb{R})$; the proofs below give the details.

A direct computation, given below, checks the biorthogonality (40) and the decomposition

$$K_N^a(x,y) = \sum_{i=0}^{2N-1} e_i^a(x) e^{\dagger a}_i(y) = \frac{1}{2}\sum_{n=0}^{N-1} \left(\phi_n^{a\prime}(x)\psi_n^a(y) - \phi_n^a(x)\psi_n^{a\prime}(y)\right). \tag{52}$$

i Details and proofs: biorthogonality and completeness

For all n, m we calculate

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}x\ \psi_n^{a\prime}(x)\phi_m^a(x) &= \frac{2}{(2\pi)^2\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}t \oint_{\pm a} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}x\ u \frac{(a^2+t^2)^m}{(a^2-u^2)^{n+1}} \mathrm{e}^{-\frac{u^2}{2}+ux-\frac{t^2}{2}-\mathrm{i}tx} \\
&= \frac{2}{(2\pi)^2\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}t \oint_{\pm a} \mathrm{d}u\ u \frac{(a^2+(t-\mathrm{i}u)^2)^m}{(a^2-u^2)^{n+1}} \mathrm{e}^{-\frac{u^2}{2}-\frac{(t-\mathrm{i}u)^2}{2}} \int_{-\infty}^{\infty} \mathrm{d}x\ \mathrm{e}^{-\mathrm{i}tx} \\
&= \frac{2}{2\pi\mathrm{i}} \oint_{\pm a} \mathrm{d}u\ u \frac{(a^2+(t-\mathrm{i}u)^2)^m}{(a^2-u^2)^{n+1}} \mathrm{e}^{-\frac{u^2}{2}-\frac{(t-\mathrm{i}u)^2}{2}} \Big|_{t=0} \\
&= \frac{2}{2\pi\mathrm{i}} \oint_{\pm a} \mathrm{d}u\ u\ (a^2-u^2)^{m-n-1} = -2\delta_{n,m},
\end{aligned} \tag{53}$$

where the second step is the change of variables $t \to t - \mathrm{i}u$. The last integral vanishes for $m > n$, where the integrand is a polynomial, and for $m < n$, where it is a derivative: $u(a^2-u^2)^{m-n-1} = -\frac{1}{2(m-n)}\partial_u (a^2-u^2)^{m-n}$. For $m = n$ both residues at $u = \pm a$ are

equal to $-\frac{1}{2}$. The identities

$$\int_{-\infty}^{\infty} \mathrm{d}x\ \psi_n^a{}'(x)\phi_m^a{}'(x) = 0, \quad \int_{-\infty}^{\infty} \mathrm{d}x\ \psi_n^a(x)\phi_m^a(x) = 0, \quad \int_{-\infty}^{\infty} \mathrm{d}x\ \psi_n^a(x)\phi_m^a{}'(x) = 2\delta_{n,m} \tag{54}$$

follow: the first two by parity, since ϕ_m^a is even and ψ_n^a is odd, and the third from the computation above by integration by parts. The completeness relation is proved as follows. At the step where we sum the geometric series in n, we enlarge, for each t, the contour around $\pm a$ to the circle $|u| = R_t$ with

$$R_t > \sqrt{t^2 + 2a^2}. \tag{55}$$

The integrals do not change, since every term is singular only at $u = \pm a$. On this circle $|a^2 - u^2| \geq R_t^2 - a^2 > a^2 + t^2$, so the series converges uniformly in u, and the circle encloses the pole $u = \mathrm{i}t$ of its sum.

$$\begin{aligned}
&\sum_{i=0}^{\infty} e_i(x)e_i^\dagger(y) = \frac{1}{2}\sum_{n=0}^{\infty}(\phi_n{}'(x)\psi_n(y) - \phi_n(x)\psi_n{}'(y)) \\
&= \frac{1}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}t\ \oint \mathrm{d}u\ \sum_{n=0}^{\infty}(-\mathrm{i}t)\frac{(a^2+t^2)^n}{(a^2-u^2)^{n+1}}\mathrm{e}^{-\frac{t^2}{2}-\mathrm{i}tx-\frac{u^2}{2}+uy} \\
&\quad - \frac{1}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}t\ \oint \mathrm{d}u\ \sum_{n=0}^{\infty}u\frac{(a^2+t^2)^n}{(a^2-u^2)^{n+1}}\mathrm{e}^{-\frac{t^2}{2}-\mathrm{i}tx-\frac{u^2}{2}+uy} \\
&= \frac{-1}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}t\ \oint \mathrm{d}u\ \sum_{n=0}^{\infty}(u+\mathrm{i}t)\frac{(a^2+t^2)^n}{(a^2-u^2)^{n+1}}\mathrm{e}^{-\frac{t^2}{2}-\mathrm{i}tx-\frac{u^2}{2}+uy} \\
&= \frac{-1}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}t\ \oint_{|u|=R_t} \mathrm{d}u\ \frac{u+\mathrm{i}t}{a^2-u^2-(a^2+t^2)}\mathrm{e}^{-\frac{t^2}{2}-\mathrm{i}tx-\frac{u^2}{2}+uy} \\
&= \frac{1}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}t\ \oint_{|u|=R_t} \mathrm{d}u\ \frac{u+\mathrm{i}t}{u^2+t^2}\mathrm{e}^{-\frac{t^2}{2}-\mathrm{i}tx-\frac{u^2}{2}+uy} \\
&= \frac{1}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}t\ \oint_{|u|=R_t} \mathrm{d}u\ \frac{1}{u-\mathrm{i}t}\mathrm{e}^{-\frac{t^2}{2}-\mathrm{i}tx-\frac{u^2}{2}+uy} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}t\ \mathrm{e}^{-\frac{t^2}{2}-\mathrm{i}tx-\frac{u^2}{2}+uy}\Big|_{u=\mathrm{i}t} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}t\ \mathrm{e}^{-\mathrm{i}tx+\mathrm{i}ty} = \delta(x-y).
\end{aligned} \tag{56}$$

This computation is formal: on the enlarged circle the factor $\mathrm{e}^{-\frac{t^2}{2}-\frac{u^2}{2}}$ does not decay as $t \to \pm\infty$, so the interchange of the sum with the integral over t and the last step are not justified as written.

In which sense, then, is this a basis? A basis of $L_2(\mathbb{R})$ would give the expansion

$$f(x) = \sum_{i=0}^{+\infty} e_i^a(x) \int_{-\infty}^{+\infty} \mathrm{d}y \; {e^\dagger}_i^a(y) f(y) \tag{57}$$

for every $f \in L_2(\mathbb{R})$. We do not claim this: the functions ${e^\dagger}_{2n}^a$ and ${e^\dagger}_{2n+1}^a$ grow like $|x|^n \mathrm{e}^{a|x|}$, so for a general f the coefficients need not even be defined. The expansion does hold, with finitely many terms, on the functions $f(x) = p(x)\,\mathrm{e}^{-\frac{x^2}{2}}$ with a polynomial p, which are dense in $L_2(\mathbb{R})$. Indeed, ϕ_n^a is an even polynomial of degree $2n$ times $\mathrm{e}^{-\frac{x^2}{2}}$ and $\phi_n^{a\prime}$ an odd one of degree $2n+1$, so $e_0^a, \ldots, e_{2k-1}^a$ span these f with $\deg p < 2k$. Writing $f = \sum_{j<2k} c_j e_j^a$, the pairings with ${e^\dagger}_i^a$ converge, since a Gaussian beats exponential growth, and (40) gives

$$\int_{-\infty}^{+\infty} \mathrm{d}y \; {e^\dagger}_i^a(y) f(y) = \begin{cases} c_i, & i < 2k, \\ 0, & i \geq 2k, \end{cases} \tag{58}$$

so the series above is the finite sum that defines f.

i Details and proofs: the kernel decomposition

The kernel decomposition needs only a finite sum, so its proof is not formal. Let the contours around $\pm a$ be circles of radius less than a. They do not enclose $u = \mathrm{i}t$, since $|\mathrm{i}t \mp a| \geq a$. By the finite geometric sum,

$$\begin{aligned}
&\sum_{i=0}^{2N-1} e_i(x) e_i^\dagger(y) = \frac{1}{2} \sum_{n=0}^{N-1} \left(\phi_n{}'(x)\psi_n(y) - \phi_n(x)\psi_n{}'(y)\right) \\
&= \frac{-1}{(2\pi)^2 \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}t \oint_{\pm a} \mathrm{d}u \; \sum_{n=0}^{N-1} (u + \mathrm{i}t) \frac{(a^2+t^2)^n}{(a^2-u^2)^{n+1}} \mathrm{e}^{-\frac{t^2}{2} - \mathrm{i}tx - \frac{u^2}{2} + uy} \\
&= \frac{1}{(2\pi)^2 \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}t \oint_{\pm a} \mathrm{d}u \; \left(1 - \left(\frac{a^2+t^2}{a^2-u^2}\right)^N\right) \frac{1}{u - \mathrm{i}t} \mathrm{e}^{-\frac{t^2}{2} - \mathrm{i}tx - \frac{u^2}{2} + uy} \\
&= \frac{-1}{(2\pi)^2 \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}t \oint_{\pm a} \mathrm{d}u \; \left(\frac{a^2+t^2}{a^2-u^2}\right)^N \frac{1}{u - \mathrm{i}t} \mathrm{e}^{-\frac{t^2}{2} - \mathrm{i}tx - \frac{u^2}{2} + uy} \\
&= K_N^a(x, y).
\end{aligned} \tag{59}$$

The term with 1 vanishes, because its integrand is analytic inside the circles around $\pm a$.

Connection with the literature

The basis above is a form of the multiple Hermite polynomials, the multiple orthogonal polynomials of the external source model (Bleher and Kuijlaars, 2004). Let P_{n_1,n_2} be the type II multiple Hermite polynomial and Q_{n_1,n_2} the type I function for the two weights $\mathrm{e}^{-\frac{x^2}{2}\pm ax}$, in the integral representations of Bleher and Kuijlaars, 2005 (eqs. (2.2) and (2.9)). Up to constant factors,

$$\phi_n^a(x) \propto \mathrm{e}^{-\frac{x^2}{2}} P_{n,n}(x), \quad \phi_n^{a\,\prime}(x) \propto \mathrm{e}^{-\frac{x^2}{2}} \left(P_{n+1,n}(x) + P_{n,n+1}(x)\right), \tag{60}$$

$$\psi_n^a(x) \propto \mathrm{e}^{\frac{x^2}{2}} Q_{n+1,n+1}(x), \quad \psi_n^{a\,\prime}(x) \propto \mathrm{e}^{\frac{x^2}{2}} \left(Q_{n+1,n}(x) + Q_{n,n+1}(x)\right). \tag{61}$$

The standard construction (eq. (1.10) there) runs along a single path of multi-indices, such as $(n,n) \to (n+1,n) \to (n+1,n+1)$. Ours takes the symmetric combination at the odd levels instead, so that every basis function is even or odd, and is conjugated by $\mathrm{e}^{\mp\frac{x^2}{2}}$, which is the freedom $K(x,y) \to \frac{f(x)}{f(y)}K(x,y)$ above. The third-order equation for ϕ_n^a is the case $n_1 = n_2 = n$ of the equation for type II multiple Hermite polynomials in Filipuk, Van Assche and Zhang (§5.1). For symmetric weights and equal indices, n enters that equation linearly, which is what makes it an eigenvalue equation.

2.3.2 Limit basis

The proofs in this section are formal computations, given for illustration; we say where each one is not justified.

We now take the analogue of the scaling limits (22). Let $N, n \to +\infty$ such that

$$\frac{2^{\frac{1}{2}}\,(N-n-1)}{N^{\frac{1}{2}}} \to \lambda \tag{62}$$

for some fixed $\lambda \in \mathbb{R}$. After the substitution $t = (2N)^{\frac{1}{4}} s$, the integrand of $\phi_n^{(2N)^{\frac{1}{2}}}$ converges pointwise:

$$(2N)^{-n}\left(t^2+2N\right)^n \mathrm{e}^{-\frac{t^2}{2}} = \mathrm{e}^{\,n\log\left(1+(2N)^{-\frac{1}{2}}s^2\right)-\frac{(2N)^{\frac{1}{2}}s^2}{2}} \quad \longrightarrow \quad \mathrm{e}^{-\frac{s^4}{4}-\frac{\lambda s^2}{2}}, \tag{63}$$

and the substitution $u = (2N)^{\frac{1}{4}} v$ does the same for $\psi_n^{(2N)^{\frac{1}{2}}}$. Passing to the limit under the integral, and for ψ_n moving the contour from the contours around $\pm a$ to $\mathcal{C}$ below, is formal here. Numerically, the ratio of the two sides of the limits below tends to 1 with an error of order $N^{-\frac{1}{2}}$. We obtain, for each fixed x,

$$2^{-\frac{1}{2}}(2N)^{-n-\frac{1}{4}}\,\phi_n^{(2N)^{\frac{1}{2}}}\left((2N)^{-\frac{1}{4}}x\right) \quad \longrightarrow \quad \phi_\lambda(x), \tag{64}$$

$$2^{-\frac{1}{2}}(2N)^{n+\frac{3}{4}}\,\psi_n^{(2N)^{\frac{1}{2}}}\left((2N)^{-\frac{1}{4}}x\right) \quad\longrightarrow\quad \psi_\lambda(x), \tag{65}$$

where

$$\phi_\lambda(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{e}^{-\frac{t^4}{4}-\frac{\lambda t^2}{2}-\mathrm{i}tx}, \tag{66}$$

$$\psi_\lambda(x) := \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}} \mathrm{d}u\, \mathrm{e}^{\frac{u^4}{4}-\frac{\lambda u^2}{2}+ux}. \tag{67}$$

Here and below, $\int_{\mathcal{C}}$ is the integral over the union of two infinite paths in the complex plane: one from $+\infty\cdot\mathrm{e}^{\mathrm{i}\frac{1}{4}\pi}$ to $+\infty\cdot\mathrm{e}^{\mathrm{i}\frac{3}{4}\pi}$, the other from $+\infty\cdot\mathrm{e}^{-\mathrm{i}\frac{3}{4}\pi}$ to $+\infty\cdot\mathrm{e}^{-\mathrm{i}\frac{1}{4}\pi}$. Up to a change of variables, ϕ_λ and ψ_λ are the Pearcey integrals of Bleher and Kuijlaars; the note after the asymptotics below gives the details.

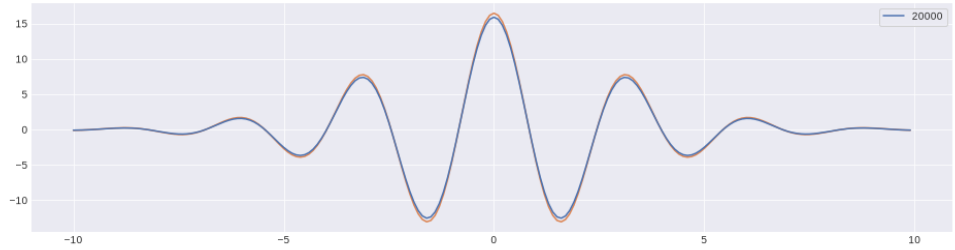

Figure 6: Illustration of the limit of ϕ_n above for $\lambda = -4$. The blue curve is the rescaled ϕ_n for the N shown in the legend, with $n = \text{round}\left(N - 1 - \lambda(N/2)^{\frac{1}{2}}\right)$, chosen so that the ratio in the condition above equals λ up to rounding; the orange curve is ϕ_λ. A negative λ gives $n > N-1$, so these functions do not enter the kernel K_N^a, and in the limit ϕ_λ does not enter K^{Pear}, which is the part $\lambda > 0$. We show them because at $\lambda < 0$ the functions oscillate and the convergence is easier to see.

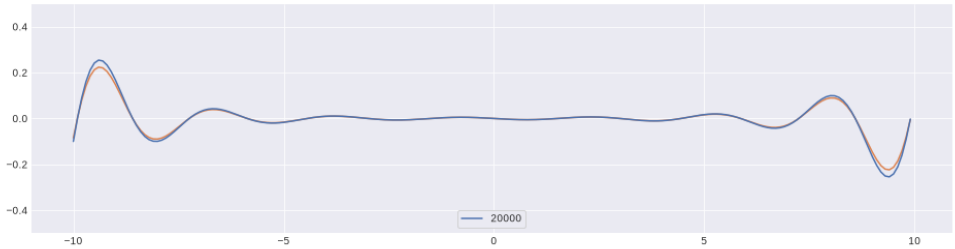

Figure 7: The same for the limit of ψ_n, with ψ_λ in orange. For small N the blue curve leaves the frame.

The functions ϕ_λ and ψ_λ solve the third-order differential eigenvalue equations

$$(\partial_x^3 - x)\,\phi_\lambda(x) = \lambda\,\partial_x\phi_\lambda(x), \tag{68}$$

$$(\partial_x^3 + x)\,\psi_\lambda(x) = \lambda\,\partial_x\psi_\lambda(x) \tag{69}$$

for $\lambda \in \mathbb{R}$.

Details and proofs: the limit equations

After the substitution $a = (2N)^{\frac{1}{2}}$, $x \to (2N)^{-\frac{1}{4}}x$, the eigenvalue equation for ϕ_n^a becomes

$$\begin{aligned}
&\left((2N)^{\frac{5}{4}}\partial_x + (2N)^{\frac{3}{4}}x - (2N)^{\frac{3}{4}}\partial_x^3 - (2N)^{\frac{1}{4}}x\partial_x^2\right)\phi_n^{(2N)^{\frac{1}{2}}}((2N)^{-\frac{1}{4}}x) = (2N)^{\frac{1}{4}}2(n+1)\partial_x\phi_n^{(2N)^{\frac{1}{2}}}((2N)^{-\frac{1}{4}}x)\\
\Leftrightarrow\quad &\left(\partial_x^3 - x + 2^{-\frac{1}{2}}N^{-\frac{1}{2}}x\partial_x^2\right)\phi_n^{(2N)^{\frac{1}{2}}}((2N)^{-\frac{1}{4}}x) = \frac{2^{\frac{1}{2}}\,(N-n-1)}{N^{\frac{1}{2}}}\partial_x\phi_n^{(2N)^{\frac{1}{2}}}\left((2N)^{-\frac{1}{4}}x\right)\\
\longrightarrow\quad &(\partial_x^3 - x)\,\phi_\lambda(x) = \lambda\partial_x\phi_\lambda(x).
\end{aligned} \tag{70}$$

For ψ_n^a we have analogously

$$\begin{aligned}
&\left((2N)^{\frac{5}{4}}\partial_x - (2N)^{\frac{3}{4}}x - (2N)^{\frac{3}{4}}\partial_x^3 + (2N)^{\frac{1}{4}}\partial_x^2 x\right)\psi_n^{(2N)^{\frac{1}{2}}}((2N)^{-\frac{1}{4}}x) = (2N)^{\frac{1}{4}}2(n+1)\partial_x\psi_n^{(2N)^{\frac{1}{2}}}((2N)^{-\frac{1}{4}}x)\\
\Leftrightarrow\quad &\left(\partial_x^3 + x - 2^{-\frac{1}{2}}N^{-\frac{1}{2}}\partial_x^2 x\right)\psi_n^{(2N)^{\frac{1}{2}}}((2N)^{-\frac{1}{4}}x) = \frac{2^{\frac{1}{2}}\,(N-n-1)}{N^{\frac{1}{2}}}\partial_x\psi_n^{(2N)^{\frac{1}{2}}}((2N)^{-\frac{1}{4}}x)\\
\longrightarrow\quad &(\partial_x^3 + x)\,\psi_\lambda(x) = \lambda\partial_x\psi_\lambda(x).
\end{aligned} \tag{71}$$

At $\lambda = 0$ they are the functions of the Pearcey kernel:

$$\phi_0(x) = \phi(x), \quad \psi_0(x) = \psi(x). \tag{72}$$

As $|x| \to \infty$ with λ fixed, steepest descent gives

$$\phi_\lambda(x) = \sqrt{\tfrac{2}{3\pi}}\,|x|^{-\frac{1}{3}}\,\mathrm{e}^{\frac{\lambda^2}{6} - \frac{3}{8}|x|^{4/3} - \frac{\lambda}{4}|x|^{2/3}}\left(\cos\left(\tfrac{3\sqrt{3}}{8}|x|^{4/3} - \tfrac{\sqrt{3}}{4}\lambda|x|^{2/3} - \tfrac{\pi}{6}\right) + O(|x|^{-2/3})\right), \tag{73}$$

$$\psi_\lambda(x) = \operatorname{sgn}(x)\sqrt{\tfrac{2}{3\pi}}\,|x|^{-\frac{1}{3}}\,\mathrm{e}^{-\frac{\lambda^2}{6} + \frac{3}{8}|x|^{4/3} + \frac{\lambda}{4}|x|^{2/3}}\left(\cos\left(\tfrac{3\sqrt{3}}{8}|x|^{4/3} - \tfrac{\sqrt{3}}{4}\lambda|x|^{2/3} + \tfrac{2\pi}{3}\right) + O(|x|^{-2/3})\right). \tag{74}$$

Connection with the literature

Up to a change of variables, ϕ_λ and ψ_λ are the Pearcey integrals p and q of Bleher and Kuijlaars (eqs. (1.6)–(1.8)) with $b = \lambda$, and the eigenvalue equations above are their third-order equations. There b labels a family of Pearcey kernels with K^{Pear} at $b = 0$, and Tracy and Widom (eq. (1.1)) use the same parameter as the time of the extended Pearcey

kernel. The asymptotics (73) and (74) follow from Lemma 8.1 of Bleher and Kuijlaars; for ϕ_λ the complete expansion is eq. (10) of López and Pagola. The decomposition of K^{Pear} at the end of this section integrates over $\lambda > 0$ the derivative of the Pearcey kernel in this parameter (Adler, Orantin and van Moerbeke, eq. (1.6)).

We now define a two-fold real continuous biorthogonal basis: for each fixed $\lambda \in \mathbb{R}$, there are two left basis functions $e_\lambda^\dagger$ and two right basis functions e_λ, one even and one odd in each pair. The real Fourier basis is the orthogonal analogue: for each $\lambda \in \mathbb{R}$, it has two real basis functions $\{\sin(\lambda x), \cos(\lambda x)\}$.

The pair of collections

$$\{-2^{-\frac{1}{2}}\psi_\lambda', 2^{-\frac{1}{2}}\psi_\lambda\}_{\lambda\in\mathbb{R}}, \quad \{2^{-\frac{1}{2}}\phi_\lambda, 2^{-\frac{1}{2}}\phi_\lambda'\}_{\lambda\in\mathbb{R}} \tag{75}$$

is a biorthogonal basis in the sense described below:

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \psi_\lambda'(x)\phi_\mu(x) = -2\,\delta(\lambda-\mu), \tag{76}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \psi_\lambda'(x)\phi_\mu'(x) = 0, \tag{77}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \psi_\lambda(x)\phi_\mu(x) = 0, \tag{78}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \psi_\lambda(x)\phi_\mu'(x) = 2\,\delta(\lambda-\mu), \tag{79}$$

and

$$\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}\lambda\, (\phi_\lambda{}'(x)\psi_\lambda(y) - \phi_\lambda(x)\psi_\lambda{}'(y)) = \delta(x-y). \tag{80}$$

Here the delta function needs more care than in the orthogonal case. By (73) and (74), the functions ϕ_λ lie in $L_2(\mathbb{R})$ and the functions ψ_λ do not. In each pairing the leading exponents $\pm\frac{3}{8}|x|^{4/3}$ cancel, and the next ones do not: for $\mu < \lambda$ the integral over x in the first identity diverges, since its integrand grows like $\mathrm{e}^{\frac{\lambda-\mu}{4}|x|^{2/3}}$. Schwartz test functions do not repair this: $\phi_\mu(0)$ grows like $\mathrm{e}^{\mu^2/4}$ as $\mu \to -\infty$, so $\int \mathrm{d}\mu\, g(\mu)\,\phi_\mu(x)$ diverges for a general Schwartz function g. A class that works is the span of the Gaussians

$$g_m(\mu) = \mathrm{e}^{-\frac{(\mu-m)^2}{3}}, \quad m \in \mathbb{R}. \tag{81}$$

Tested against them, the four identities hold; the first one, for instance, reads

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \psi'_\lambda(x) \int_{-\infty}^{+\infty} \mathrm{d}\mu\, g_m(\mu)\, \phi_\mu(x) = -2\, g_m(\lambda). \tag{82}$$

The completeness relation holds in the form

$$f(x) = \lim_{\Lambda\to+\infty} \int_{-\infty}^{+\infty} \mathrm{d}y\, f(y)\, \frac{1}{2} \int_{-\Lambda}^{+\infty} \mathrm{d}\lambda\, (\phi'_\lambda(x)\psi_\lambda(y) - \phi_\lambda(x)\psi'_\lambda(y)) \tag{83}$$

for continuously differentiable f with compact support; Schwartz functions are again too many, since ψ_λ grows. We prove it only formally.

i Details and proofs

For the Gaussian g_m the integral over μ is Gaussian too:

$$G_m(x) := \int_{-\infty}^{+\infty} \mathrm{d}\mu\, g_m(\mu)\, \phi_\mu(x) = \sqrt{6\pi}\, \phi_{2m}(\sqrt{2}\, x). \tag{84}$$

By (73) it decays like $\mathrm{e}^{-\frac{3}{8}2^{2/3}|x|^{4/3}}$, fast enough to integrate over x before u, with $\mathcal{C}$ the rays through the origin:

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \psi'_\lambda(x)\, G_m(x) = \frac{\sqrt{3\pi}}{2\pi\mathrm{i}} \int_{\mathcal{C}} \mathrm{d}u\, u\, \mathrm{e}^{\frac{3u^4}{16} + \frac{(m-\lambda)u^2}{2}} = -2\, g_m(\lambda), \tag{85}$$

a Gaussian integral in $\frac{u^2}{2} \in \mathrm{i}\mathbb{R}$. In the other three identities, with G'_m in place of G_m where ϕ'_μ appears, the factor u becomes $-u^2$, 1 and $-u$; the first two integrals vanish, since $u \mapsto -u$ maps $\mathcal{C}$ onto itself with its orientation. The span of the g_m is dense in $L_2(\mathbb{R})$ and in the Schwartz space, so these test functions determine the delta function uniquely.

The proof of the last identity is formal: it changes the order of integration, and two more steps are not justified. Here $\mathcal{C}$ consists of the rays $\arg u = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ through the origin, since the factor $(u - \mathrm{i}t)^{-1}$ has poles on the imaginary axis. The integral over λ diverges at $\lambda \to -\infty$, so we cut it at $-\Lambda$ and let $\Lambda \to +\infty$ at the end, and we do not prove that

the remaining contour integral vanishes in this limit:

$$
\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}\lambda\,(\phi_\lambda{}'(x)\psi_\lambda(y)-\phi_\lambda(x)\psi_\lambda{}'(y))\\
&=\int_{-\infty}^{+\infty}\mathrm{d}\lambda\frac{1}{2(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}}\mathrm{d}u\,(-\mathrm{i}t-u)\mathrm{e}^{-\frac{t^4}{4}-\frac{\lambda t^2}{2}-\mathrm{i}tx+\frac{u^4}{4}-\frac{\lambda u^2}{2}+uy}\\
&=\frac{-1}{2(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}}\mathrm{d}u\int_{-\infty}^{+\infty}\mathrm{d}\lambda\frac{u^2+t^2}{u-\mathrm{i}t}\mathrm{e}^{-\frac{t^4}{4}-\frac{\lambda t^2}{2}-\mathrm{i}tx+\frac{u^4}{4}-\frac{\lambda u^2}{2}+uy}\\
&=-\frac{\lim_{\Lambda\to+\infty}}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}}\mathrm{d}u\int_{-\Lambda}^{+\infty}\mathrm{d}\lambda\frac{-\partial_\lambda}{u-\mathrm{i}t}\mathrm{e}^{-\frac{t^4}{4}-\frac{\lambda t^2}{2}-\mathrm{i}tx+\frac{u^4}{4}-\frac{\lambda u^2}{2}+uy} \qquad (86)\\
&=-\frac{\lim_{\Lambda\to+\infty}}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}}\mathrm{d}u\,\frac{\mathrm{e}^{-\frac{t^4}{4}-\mathrm{i}tx+\frac{u^4}{4}+uy+\frac{\Lambda}{2}(t^2+u^2)}}{u-\mathrm{i}t}\\
&=\frac{\lim_{\Lambda\to+\infty}}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}t\;\mathrm{e}^{-\frac{t^4}{4}-\mathrm{i}tx+\frac{u^4}{4}+uy+\frac{\Lambda}{2}(t^2+u^2)}\Big|_{u=\mathrm{i}t}\\
&\quad-\frac{\lim_{\Lambda\to+\infty}}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}'}\mathrm{d}u\,\frac{\mathrm{e}^{-\frac{t^4}{4}-\mathrm{i}tx+\frac{u^4}{4}+uy+\frac{\Lambda}{2}(t^2+u^2)}}{u-\mathrm{i}t}.
\end{aligned}
$$

The first term is the residue at $u=\mathrm{i}t$, picked up when $\mathcal{C}$ is deformed across the pole into a contour $\mathcal{C}'$ on which $\mathrm{Re}\,(t^2+u^2)<0$. With the orientation of $\mathcal{C}$ the pole lies to the right of the path, so the residue enters with the factor $-2\pi\mathrm{i}$. We assume, without proof, that the integral over $\mathcal{C}'$ tends to zero as $\Lambda\to+\infty$. Then

$$
\ldots=\frac{\lim_{\Lambda\to+\infty}}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}t\,\mathrm{e}^{-\mathrm{i}tx+\mathrm{i}ty+\frac{\Lambda}{2}\cdot 0}=\delta(x-y). \qquad (87)
$$

The Pearcey kernel has the decomposition

$$
K^{\mathrm{Pear}}(x,y)=\frac{1}{2}\int_{0}^{+\infty}\mathrm{d}\lambda\,(\phi'_\lambda(x)\psi_\lambda(y)-\phi_\lambda(x)\psi'_\lambda(y))\,. \qquad (88)
$$

Details and proofs

We compute the right-hand side with the opposite sign, with $\mathcal{C}$ the rays through the origin, as in the proof of completeness above.

$$
\begin{aligned}
&\frac{1}{2}\int_0^{+\infty} \mathrm{d}\lambda\,(\phi_\lambda(x)\psi'_\lambda(y)-\phi'_\lambda(x)\psi_\lambda(y))\\
&=\int_0^{+\infty}\mathrm{d}\lambda\frac{1}{2(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}}\mathrm{d}u\,(u+\mathrm{i}t)\mathrm{e}^{-\frac{t^4}{4}-\frac{\lambda t^2}{2}-\mathrm{i}tx+\frac{u^4}{4}-\frac{\lambda u^2}{2}+uy}\\
&=\frac{1}{2(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}}\mathrm{d}u\int_0^{+\infty}\mathrm{d}\lambda\frac{u^2+t^2}{u-\mathrm{i}t}\mathrm{e}^{-\frac{t^4}{4}-\frac{\lambda t^2}{2}-\mathrm{i}tx+\frac{u^4}{4}-\frac{\lambda u^2}{2}+uy}\\
&=\frac{1}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}}\mathrm{d}u\int_0^{+\infty}\mathrm{d}\lambda\frac{-\partial_\lambda}{u-\mathrm{i}t}\mathrm{e}^{-\frac{t^4}{4}-\frac{\lambda t^2}{2}-\mathrm{i}tx+\frac{u^4}{4}-\frac{\lambda u^2}{2}+uy}\\
&=\frac{1}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}}\mathrm{d}u\,\frac{\mathrm{e}^{-\frac{t^4}{4}-\mathrm{i}tx+\frac{u^4}{4}+uy}}{u-\mathrm{i}t}\\
&=\frac{1}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}}\mathrm{d}u\,\frac{\mathrm{e}^{-\frac{t^4}{4}+\frac{u^4}{4}}}{(u-\mathrm{i}t)(x-y)}(\mathrm{i}\partial_t-\partial_u)\mathrm{e}^{-\mathrm{i}tx+uy} \qquad (89)\\
&=-\frac{1}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}}\mathrm{d}u\,\mathrm{e}^{-\mathrm{i}tx+uy}(\mathrm{i}\partial_t-\partial_u)\frac{\mathrm{e}^{-\frac{t^4}{4}+\frac{u^4}{4}}}{(u-\mathrm{i}t)(x-y)}\\
&=-\frac{1}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}}\mathrm{d}u\,\mathrm{e}^{-\mathrm{i}tx+uy}\frac{(-\mathrm{i}t^3-u^3)\mathrm{e}^{-\frac{t^4}{4}+\frac{u^4}{4}}}{(u-\mathrm{i}t)(x-y)}\\
&=-\frac{1}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}}\mathrm{d}u\,\frac{(t^2-\mathrm{i}tu-u^2)\mathrm{e}^{-\frac{t^4}{4}+\frac{u^4}{4}-\mathrm{i}tx+uy}}{x-y}\\
&=-\frac{1}{(2\pi)^2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{\mathcal{C}}\mathrm{d}u\,\frac{(-\partial_x^2+\partial_x\partial_y-\partial_y^2)\mathrm{e}^{-\frac{t^4}{4}+\frac{u^4}{4}-\mathrm{i}tx+uy}}{x-y}\\
&=-\frac{\phi'(x)\psi'(y)-\phi''(x)\psi(y)-\phi(x)\psi''(y)}{x-y}=-K(x,y).
\end{aligned}
$$

This computation is formal. The integral over λ is exchanged with the double integral without absolute convergence, the boundary term at $\lambda=+\infty$ is dropped, and the integration by parts passes the pole $u=\mathrm{i}t$ without justification.

As in the Airy kernel decomposition (27), the kernel projects onto the positive half of the spectrum.

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